

Symbol	Definition
α	Smoothing parameter for the level of the series
γ	Smoothing parameter for trend
δ	Smoothing parameter for seasonal factors
ϕ	Trend modification parameter
β	Discount factor, $0 \leq \beta \leq 1$
S_t	Smoothed level of the series, computed after X_t is observed. Also the expected value of the data at the end of period t in some models
T_t	Smoothed trend at the end of period t . Can be additive (linear) or multiplicative
I_t	Smoothed seasonal index or factor at the end of period t . Can be additive or multiplicative
S_t''	Double-smoothed average (from an application of simple exponential smoothing to S_t)
X_t	Observed value of the time series in period t
m	Number of periods in the forecast lead-time
p	Number of periods in the seasonal cycle
$\hat{X}_t(m)$	Forecast for m periods ahead from origin t
e_t	One-period-ahead forecast error, $e_t = X_t - \hat{X}_{t-1}(1)$. Note that $e_t(m)$ should be used for other forecast origins.

Exhibit 2. Standard notation for exponential smoothing

Thus model components and parameters have some intuitive meaning to the user. Only limited data storage and computational effort are required. Tracking signal tests for forecast control are easy to apply.

Perhaps the most important reason for the popularity of exponential smoothing is the surprising accuracy that can be obtained with minimal effort in model identification. Two large-scale empirical studies have found little difference in forecast accuracy between exponential smoothing and ARIMA models identified by the Box-Jenkins (1976) methodology—see Makridakis and Hibon (1979) and Makridakis *et al.* (1982).

Despite the large body of research on exponential smoothing, there has never been a comprehensive review of the subject. This paper reviews the research since the original work by Brown and Holt in the 1950s. Sections 1–3 discuss the class of relatively simple models which rely on the Holt-Winters heuristic decomposition procedure for seasonal data. These models are appropriate in inventory control systems, when the noise component of the time series is relatively large, and when limited historical data rule out more sophisticated models.

The discussion of Holt-Winters is illustrated by Exhibits 1–7. Exhibit 1 shows examples of forecast profiles. There is no agreement in the literature on notation for Holt-Winters so Exhibit 2

Model	Recurrence form	Error-correction form
3-1 Non-seasonal	$S_t = \alpha X_t + (1 - \alpha)S_{t-1}$ $\hat{X}_t(m) = S_t$	$S_t = S_{t-1} + \alpha e_t$ $\hat{X}_t(m) = S_t$
3-2 Additive seasonals	$S_t = \alpha(X_t - I_{t-p}) + (1 - \alpha)S_{t-1}$ $I_t = \delta(X_t - S_t) + (1 - \delta)I_{t-p}$ $\hat{X}_t(m) = S_t + I_{t-p+m}$	$S_t = S_{t-1} + \alpha e_t$ $I_t = I_{t-p} + \delta(1 - \alpha)e_t$ $\hat{X}_t(m) = S_t + I_{t-p+m}$
3-3 Multiplicative seasonals	$S_t = \alpha(X_t/I_{t-p}) + (1 - \alpha)S_{t-1}$ $I_t = \delta(X_t/S_t) + (1 - \delta)I_{t-p}$ $\hat{X}_t(m) = S_t I_{t-p+m}$	$S_t = S_{t-1} + \alpha e_t/I_{t-p}$ $I_t = I_{t-p} + \delta(1 - \alpha)e_t/S_t$ $\hat{X}_t(m) = S_t I_{t-p+m}$

Exhibit 3. Constant level models (simple smoothing)

Model	Recurrence form	Error-correction form
4-1 Non-seasonal	$S_t = \alpha X_t + (1 - \alpha)(S_{t-1} + T_{t-1})$ $T_t = \gamma(S_t - S_{t-1}) + (1 - \gamma)T_{t-1}$ $\hat{X}_t(m) = S_t + mT_t$	$S_t = S_{t-1} + T_{t-1} + \alpha e_t$ $T_t = T_{t-1} + \alpha \gamma e_t$ $\hat{X}_t(m) = S_t + mT_t$
4-2 Additive seasonals	$S_t = \alpha(X_t - I_{t-p}) + (1 - \alpha)(S_{t-1} + T_{t-1})$ $T_t = \gamma(S_t - S_{t-1}) + (1 - \gamma)T_{t-1}$ $I_t = \delta(X_t - S_t) + (1 - \delta)I_{t-p}$ $\hat{X}_t(m) = S_t + mT_t + I_{t-p+m}$	$S_t = S_{t-1} + T_{t-1} + \alpha e_t$ $T_t = T_{t-1} + \alpha \gamma e_t$ $I_t = I_{t-p} + \delta(1 - \alpha)e_t$ $\hat{X}_t(m) = S_t + mT_t + I_{t-p+m}$
4-3 Multiplicative seasonals	$S_t = \alpha(X_t/I_{t-p}) + (1 - \alpha)(S_{t-1} + T_{t-1})$ $T_t = \gamma(S_t - S_{t-1}) + (1 - \gamma)T_{t-1}$ $I_t = \delta(X_t/S_t) + (1 - \delta)I_{t-p}$ $\hat{X}_t(m) = (S_t + mT_t)I_{t-p+m}$	$S_t = S_{t-1} + T_{t-1} + \alpha e_t/I_{t-p}$ $T_t = T_{t-1} + \alpha \gamma e_t/I_{t-p}$ $I_t = I_{t-p} + \delta(1 - \alpha)e_t/S_t$ $\hat{X}_t(m) = (S_t + mT_t)I_{t-p+m}$

Exhibit 4. Holt-Winters linear trend models

is proposed as a standard. Exhibits 3-7 contain model formulations corresponding to the forecast profiles in Exhibit 1. With a few exceptions, each model is written in two forms, a recurrence form and an error-correction form. The recurrence forms were used in the original work by Brown and Holt and still have pedagogic value. However, the error-correction forms are equivalent and generally easier to use.

Section 1 discusses simple smoothing (Exhibit 3) for a constant-level process. Section 2 discusses models for linear trends (Exhibits 4 and 5). Section 3 deals with non-linear trends (Exhibits 6 and 7). The research in each section is reviewed according to the following questions. What are the useful properties of each model? What parameters are recommended? How should the model be initialized?

Model	Recurrence form	Error-correction form
5-1* Non-seasonal	$S_t = \alpha X_t + (1 - \alpha)S_{t-1}$ $T_t = \alpha(S_t - S_{t-1}) + (1 - \alpha)T_{t-1}$ $\hat{X}_t(m) = S_t + \left(\frac{1 - \alpha}{\alpha}\right)T_t + mT_t$	$S_t = S_{t-1} + T_{t-1} + \alpha e_t$ $T_t = T_{t-1} + \alpha^2 e_t$ $\hat{X}_t(m) = S_t + \left(\frac{1 - \alpha}{\alpha}\right)T_t + mT_t$
5-2* Non-seasonal	$S_t = \alpha X_t + (1 - \alpha)S_{t-1}$ $S_t'' = \alpha S_t + (1 - \alpha)S_{t-1}''$ $\hat{X}_t(m) = 2S_t - S_t'' + m\left(\frac{\alpha}{1 - \alpha}\right)(S_t - S_t'')$	$S_t = S_{t-1} + T_{t-1} + \alpha(2 - \alpha)e_t$ $T_t = T_{t-1} + \alpha^2 e_t$ $\hat{X}_t(m) = S_t + mT_t$
5-3 Additive seasonals	N/A	$S_t = S_{t-1} + T_{t-1} + \alpha(2 - \alpha)e_t$ $T_t = T_{t-1} + \alpha^2 e_t$ $I_t = I_{t-p} + \delta[1 - \alpha(2 - \alpha)]e_t$ $\hat{X}_t(m) = S_t + mT_t + I_{t-p+m}$
5-4 Multiplicative seasonals	N/A	$S_t = S_{t-1} + T_{t-1} + \alpha(2 - \alpha)e_t/I_{t-p}$ $T_t = T_{t-1} + \alpha^2 e_t/I_{t-p}$ $I_t = I_{t-p} + \delta[1 - \alpha(2 - \alpha)]e_t/S_t$ $\hat{X}_t(m) = (S_t + mT_t)I_{t-p+m}$

N/A, Not applicable.

Exhibit 5. Brown's linear trend models (* denotes equivalent models)

Model	Recurrence form	Error-correction form
6-1 Non-seasonal	$S_t = \alpha X_t + (1 - \alpha)(S_{t-1} T_{t-1})$ $T_t = \gamma(S_t/S_{t-1}) + (1 - \gamma)T_{t-1}$ $\hat{X}_t(m) = S_t T_t^m$	$S_t = S_{t-1} T_{t-1} + \alpha e_t$ $T_t = T_{t-1} + \alpha \gamma e_t / S_{t-1}$ $\hat{X}_t(m) = S_t T_t^m$
6-2 Additive seasonals	$S_t = \alpha(X_t - I_{t-p}) + (1 - \alpha)S_{t-1} T_{t-1}$ $T_t = \gamma(S_t/S_{t-1}) + (1 - \gamma)T_{t-1}$ $I_t = \delta(X_t - S_t) + (1 - \delta)I_{t-p}$ $\hat{X}_t(m) = S_t T_t^m + I_{t-p+m}$	$S_t = S_{t-1} T_{t-1} + \alpha e_t$ $T_t = T_{t-1} + \alpha \gamma e_t / S_{t-1}$ $I_t = I_{t-p} + \delta(1 - \alpha)e_t$ $\hat{X}_t(m) = S_t T_t^m + I_{t-p+m}$
6-3 Multiplicative seasonals	$S_t = \alpha(X_t/I_{t-p}) + (1 - \alpha)S_{t-1} T_{t-1}$ $T_t = \gamma(S_t/S_{t-1}) + (1 - \gamma)T_{t-1}$ $I_t = \delta(X_t/S_t) + (1 - \delta)I_{t-p}$ $\hat{X}_t(m) = (S_t T_t^m) I_{t-p+m}$	$S_t = S_{t-1} T_{t-1} + \alpha e_t / I_{t-p}$ $T_t = T_{t-1} + (\alpha \gamma e_t / S_{t-1}) / I_{t-p}$ $I_t = I_{t-p} + \delta(1 - \alpha)e_t / S_t$ $\hat{X}_t(m) = (S_t T_t^m) I_{t-p+m}$

Exhibit 6. Exponential trend models

In Section 4, we review the general exponential smoothing methodology according to the same questions. General exponential smoothing differs from Holt-Winters in that Fourier functions of time are used to model seasonality. This introduces a considerable mathematical complexity which has been an obstacle in practical applications. However, recent research has done much to simplify general exponential smoothing.

In Sections 5-7, we turn to problems in the maintenance of forecasting systems based on exponential smoothing. These problems apply to both Holt-Winters and general exponential

Model	Recurrence form	Error-correction form
7-1 Non-seasonal	$S_t = \alpha X_t + (1 - \alpha)(S_{t-1} + \phi T_{t-1})$ $T_t = \gamma(S_t - S_{t-1}) + (1 - \gamma)\phi T_{t-1}$ $\hat{X}_t(m) = S_t + \sum_{i=1}^m \phi^i T_t$	$S_t = S_{t-1} + \phi T_{t-1} + \alpha e_t$ $T_t = \phi T_{t-1} + \alpha \gamma e_t$ $\hat{X}_t(m) = S_t + \sum_{i=1}^m \phi^i T_t$
7-2 Non-seasonal	N/A	$S_t = S_{t-1} + \phi T_{t-1} + \alpha(2 - \alpha)e_t$ $T_t = \phi T_{t-1} + \alpha(\alpha - \phi + 1)e_t$ $\hat{X}_t(m) = S_t + \sum_{i=1}^m \phi^i T_t$
7-3 Additive seasonals	N/A	$S_t = S_{t-1} + \phi T_{t-1} + \alpha(2 - \alpha)e_t$ $T_t = \phi T_{t-1} + \alpha(\alpha - \phi + 1)e_t$ $I_t = I_{t-p} + \delta[1 - \alpha(2 - \alpha)]e_t$ $\hat{X}_t(m) = S_t + \sum_{i=1}^m \phi^i T_t + I_{t-p+m}$
7-4 Multiplicative seasonals	N/A	$S_t = S_{t-1} + \phi T_{t-1} + \alpha(2 - \alpha)e_t / I_{t-p}$ $T_t = \phi T_{t-1} + \alpha(\alpha - \phi + 1)e_t / I_{t-p}$ $I_t = I_{t-p} + \delta[1 - \alpha(2 - \alpha)]e_t / S_t$ $\hat{X}_t(m) = \left(S_t + \sum_{i=1}^m \phi^i T_t \right) I_{t-p+m}$

N/A, Not applicable.

Exhibit 7. Damped trend models ($0 < \phi < 1$)

Table 2. Notation for exponential smoothing

Symbol	Definition
α	Smoothing parameter for the level of the series
γ	Smoothing parameter for the trend
δ	Smoothing parameter for seasonal indices
ϕ	Autoregressive or damping parameter
β	Discount factor, $0 \leq \beta \leq 1$
S_t	Smoothed level of the series, computed after X_t is observed. Also the expected value of the data at the end of period t in some models
T_t	Smoothed additive trend at the end of period t
R_t	Smoothed multiplicative trend at the end of period t
I_t	Smoothed seasonal index at the end of period t . Can be additive or multiplicative
X_t	Observed value of the time series in period t
m	Number of periods in the forecast lead-time
p	Number of periods in the seasonal cycle
$\hat{X}_t(m)$	Forecast for m periods ahead from origin t
e_t	One-step-ahead forecast error, $e_t = X_t - \hat{X}_{t-1}(1)$. Note that $e_t(m)$ should be used for other forecast origins
C_t	Cumulative renormalization factor for seasonal indices. Can be additive or multiplicative
V_t	Transition variable in smooth transition exponential smoothing
D_t	Observed value of nonzero demand in the Croston method
Q_t	Observed inter-arrival time of transactions in the Croston method
Z_t	Smoothed nonzero demand in the Croston method
P_t	Smoothed inter-arrival time in the Croston method
Y_t	Estimated demand per unit time in the Croston method (Z_t/P_t)

Table 1
Standard exponential smoothing methods

Trend	Seasonality		
	N None	A Additive	M Multiplicative
N None	$S_t = \alpha X_t + (1 - \alpha)S_{t-1}$ $\hat{X}_t(m) = S_t$	$S_t = \alpha(X_t - I_{t-p}) + (1 - \alpha)S_{t-1}$ $I_t = \delta(X_t - S_t) + (1 - \delta)I_{t-p}$ $\hat{X}_t(m) = S_t + I_{t-p+m}$	$S_t = \alpha(X_t / I_{t-p}) + (1 - \alpha)S_{t-1}$ $I_t = \delta(X_t / S_t) + (1 - \delta)I_{t-p}$ $\hat{X}_t(m) = S_t I_{t-p+m}$
	$S_t = S_{t-1} + \alpha e_t$ $\hat{X}_t(m) = S_t$	$S_t = S_{t-1} + \alpha e_t$ $I_t = I_{t-p} + \delta(1 - \alpha)e_t$ $\hat{X}_t(m) = S_t + I_{t-p+m}$	$S_t = S_{t-1} + \alpha e_t / I_{t-p}$ $I_t = I_{t-p} + \delta(1 - \alpha)e_t / S_t$ $\hat{X}_t(m) = S_t I_{t-p+m}$
A Additive	$S_t = \alpha X_t + (1 - \alpha)(S_{t-1} + T_{t-1})$ $T_t = \gamma(S_t - S_{t-1}) + (1 - \gamma)T_{t-1}$ $\hat{X}_t(m) = S_t + mT_t$	$S_t = \alpha(X_t - I_{t-p}) + (1 - \alpha)(S_{t-1} + T_{t-1})$ $T_t = \gamma(S_t - S_{t-1}) + (1 - \gamma)T_{t-1}$ $I_t = \delta(X_t - S_t) + (1 - \delta)I_{t-p}$ $\hat{X}_t(m) = S_t + mT_t + I_{t-p+m}$	$S_t = \alpha(X_t / I_{t-p}) + (1 - \alpha)(S_{t-1} + T_{t-1})$ $T_t = \gamma(S_t - S_{t-1}) + (1 - \gamma)T_{t-1}$ $I_t = \delta(X_t / S_t) + (1 - \delta)I_{t-p}$ $\hat{X}_t(m) = (S_t + mT_t)I_{t-p+m}$
	$S_t = S_{t-1} + T_{t-1} + \alpha e_t$ $T_t = T_{t-1} + \alpha \gamma e_t$ $\hat{X}_t(m) = S_t + mT_t$	$S_t = S_{t-1} + T_{t-1} + \alpha e_t$ $T_t = T_{t-1} + \alpha \gamma e_t$ $I_t = I_{t-p} + \delta(1 - \alpha)e_t$ $\hat{X}_t(m) = S_t + mT_t + I_{t-p+m}$	$S_t = S_{t-1} + T_{t-1} + \alpha e_t / I_{t-p}$ $T_t = T_{t-1} + \alpha \gamma e_t / I_{t-p}$ $I_t = I_{t-p} + \delta(1 - \alpha)e_t / S_t$ $\hat{X}_t(m) = (S_t + mT_t)I_{t-p+m}$
DA Damped Additive	$S_t = \alpha X_t + (1 - \alpha)(S_{t-1} + \phi T_{t-1})$ $T_t = \gamma(S_t - S_{t-1}) + (1 - \gamma)\phi T_{t-1}$ $\hat{X}_t(m) = S_t + \sum_{i=1}^m \phi^i T_t$	$S_t = \alpha(X_t - I_{t-p}) + (1 - \alpha)(S_{t-1} + \phi T_{t-1})$ $T_t = \gamma(S_t - S_{t-1}) + (1 - \gamma)\phi T_{t-1}$ $I_t = \delta(X_t - S_t) + (1 - \delta)I_{t-p}$ $\hat{X}_t(m) = S_t + \sum_{i=1}^m \phi^i T_t + I_{t-p+m}$	$S_t = \alpha(X_t / I_{t-p}) + (1 - \alpha)(S_{t-1} + \phi T_{t-1})$ $T_t = \gamma(S_t - S_{t-1}) + (1 - \gamma)\phi T_{t-1}$ $I_t = \delta(X_t / S_t) + (1 - \delta)I_{t-p}$ $\hat{X}_t(m) = (S_t + \sum_{i=1}^m \phi^i T_t)I_{t-p+m}$
	$S_t = S_{t-1} + \phi T_{t-1} + \alpha e_t$ $T_t = \phi T_{t-1} + \alpha \gamma e_t$ $\hat{X}_t(m) = S_t + \sum_{i=1}^m \phi^i T_t$	$S_t = S_{t-1} + \phi T_{t-1} + \alpha e_t$ $T_t = \phi T_{t-1} + \alpha \gamma e_t$ $I_t = I_{t-p} + \delta(1 - \alpha)e_t$ $\hat{X}_t(m) = S_t + \sum_{i=1}^m \phi^i T_t + I_{t-p+m}$	$S_t = S_{t-1} + \phi T_{t-1} + \alpha e_t / I_{t-p}$ $T_t = \phi T_{t-1} + \alpha \gamma e_t / I_{t-p}$ $I_t = I_{t-p} + \delta(1 - \alpha)e_t / S_t$ $\hat{X}_t(m) = (S_t + \sum_{i=1}^m \phi^i T_t)I_{t-p+m}$
M Multiplicative	$S_t = \alpha X_t + (1 - \alpha)(S_{t-1} R_{t-1})$ $R_t = \gamma(S_t / S_{t-1}) + (1 - \gamma)R_{t-1}$ $\hat{X}_t(m) = S_t R_t^m$	$S_t = \alpha(X_t - I_{t-p}) + (1 - \alpha)S_{t-1} R_{t-1}$ $R_t = \gamma(S_t / S_{t-1}) + (1 - \gamma)R_{t-1}$ $I_t = \delta(X_t - S_t) + (1 - \delta)I_{t-p}$ $\hat{X}_t(m) = S_t R_t^m + I_{t-p+m}$	$S_t = \alpha(X_t / I_{t-p}) + (1 - \alpha)S_{t-1} R_{t-1}$ $R_t = \gamma(S_t / S_{t-1}) + (1 - \gamma)R_{t-1}$ $I_t = \delta(X_t / S_t) + (1 - \delta)I_{t-p}$ $\hat{X}_t(m) = (S_t R_t^m)I_{t-p+m}$
	$S_t = S_{t-1} R_{t-1} + \alpha e_t$ $R_t = R_{t-1} + \alpha \gamma e_t / S_{t-1}$ $\hat{X}_t(m) = S_t R_t^m$	$S_t = S_{t-1} R_{t-1} + \alpha e_t$ $R_t = R_{t-1} + \alpha \gamma e_t / S_{t-1}$ $I_t = I_{t-p} + \delta(1 - \alpha)e_t$ $\hat{X}_t(m) = S_t R_t^m + I_{t-p+m}$	$S_t = S_{t-1} R_{t-1} + \alpha e_t / I_{t-p}$ $R_t = R_{t-1} + (\alpha \gamma e_t / S_{t-1}) / I_{t-p}$ $I_t = I_{t-p} + \delta(1 - \alpha)e_t / S_t$ $\hat{X}_t(m) = (S_t R_t^m)I_{t-p+m}$
DM Damped Multiplicative	$S_t = \alpha X_t + (1 - \alpha)(S_{t-1} R_{t-1}^\phi)$ $R_t = \gamma(S_t / S_{t-1}) + (1 - \gamma)R_{t-1}^\phi$ $\hat{X}_t(m) = S_t R_t^{\sum_{i=1}^m \phi^i}$	$S_t = \alpha(X_t - I_{t-p}) + (1 - \alpha)S_{t-1} R_{t-1}^\phi$ $R_t = \gamma(S_t / S_{t-1}) + (1 - \gamma)R_{t-1}^\phi$ $I_t = \delta(X_t - S_t) + (1 - \delta)I_{t-p}$ $\hat{X}_t(m) = S_t R_t^{\sum_{i=1}^m \phi^i} + I_{t-p+m}$	$S_t = \alpha(X_t / I_{t-p}) + (1 - \alpha)(S_{t-1} R_{t-1}^\phi)$ $R_t = \gamma(S_t / S_{t-1}) + (1 - \gamma)R_{t-1}^\phi$ $I_t = \delta(X_t / S_t) + (1 - \delta)I_{t-1}$ $\hat{X}_t(m) = (S_t R_t^{\sum_{i=1}^m \phi^i})I_{t-p+m}$
	$S_t = S_{t-1} R_{t-1}^\phi + \alpha e_t$ $R_t = R_{t-1}^\phi + \alpha \gamma e_t / S_{t-1}$ $\hat{X}_t(m) = S_t R_t^{\sum_{i=1}^m \phi^i}$	$S_t = S_{t-1} R_{t-1}^\phi + \alpha e_t$ $R_t = R_{t-1}^\phi + \alpha \gamma e_t / S_{t-1}$ $I_t = I_{t-p} + \delta(1 - \alpha)e_t$ $\hat{X}_t(m) = S_t R_t^{\sum_{i=1}^m \phi^i} + I_{t-p+m}$	$S_t = S_{t-1} R_{t-1}^\phi + \alpha e_t / I_{t-p}$ $R_t = R_{t-1}^\phi + (\alpha \gamma e_t / S_{t-1}) / I_{t-p}$ $I_t = I_{t-p} + \delta(1 - \alpha)e_t / S_t$ $\hat{X}_t(m) = (S_t R_t^{\sum_{i=1}^m \phi^i})I_{t-p+m}$